Pergamon

Appl. Math. Lett. Vol. 9, No. 3, pp. 57–61, 1996

0893-9659/96 $15.00 + 0.00

S0893-9659(96)00032-8

Boundary Value Problems for Functional Difference Equations

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(Received and accepted September 1995)

Abstract—Solutions are obtained for the functional difference equation, $\Delta^2 x(t-1) = f(t, x_{t-1}, \Delta x(t-1))$, $1 \le t \le T+1$, satisfying the initial condition, $x(s) = \phi(s)$, $-N \le s \le 0$, and the boundary condition, $x(T+2) = A$. The methods involve the Leray-Schauder Alternative.

Keywords—Boundary value problem, Functional difference equation, Nonlinear alternative.

1. INTRODUCTION

Given integers $a < b$, we shall denote discrete sets by interval notation such as $[a,b] = \{a, a+1, \dots, b\}$, $[a,b) = \{a, \dots, b-1\}$, etc. Also, given a function $z : [a,b] \to \mathbb{R}$, define its norm, $\|z\|_{[a,b]}$, by

$$\|z\|_{[a,b]} = \max_{a \le s \le b} |z(s)|.$$

Let T and N be given natural numbers with $T > N$. Define the space of functions, $\mathcal{F}_N$, by

$$\mathcal{F}_N = \{x \mid x : [-N, 0] \to \mathbb{R}\}.$$

Then $(\mathcal{F}_N, \|\cdot\|_{[-N,0]})$ is a Banach space. Also, for each $z : [-N, T+2] \to \mathbb{R}$ and for each $t \in [0, T+2]$, let $z_t \in \mathcal{F}_N$ be defined by

$$z_t(s) = z(t+s), \qquad s \in [-N, 0].$$

In this light, we are concerned with the existence of solutions of boundary value problems for the second-order functional difference equation,

$$\Delta^2 x(t-1) = f(t, x_{t-1}, \Delta x(t-1)), \qquad t \in [1, T+1], \tag{1}$$

satisfying

$$x(s) = \phi(s), \qquad s \in [-N, 0], \tag{2}$$

$$x(T+2) = A, \tag{3}$$

where $f : [1, T+1] \times \mathcal{F}_N \times \mathbb{R} \to \mathbb{R}$ is continuous, $\phi \in \mathcal{F}_N$, and $A \in \mathbb{R}$.

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Boundary value problems for functional difference equations recently have received substantial attention by [1–5] to name a few. Some of these papers are devoted to results which arise as analogues of works on boundary value problems for functional differential equations such as [6–9]. A method which is becoming common in obtaining solutions of boundary value problems for functional differential equations involves the Topological Transversality Theorem due to Granas [10,11], and whose application can be found in [8,9,12–18].

In this paper, we apply the topological transversality in a form based on the nonlinear alternative of Leary and Schauder [10,19], to obtain solutions of (1)–(3). This is carried out in Section 2, and an example is presented. In Section 3, generalizations are presented, whose proofs are completely analogous to those for (1)–(3). As such, their proofs will be omitted.

2. LERAY-SCHAUDER ALTERNATIVE AND (1)–(3)

In this section, let $A \in \mathbb{R}$ and $\phi : [-N, 0] \to \mathbb{R}$ be given. We establish solutions of (1)–(3) as a consequence of the Leray-Schauder Nonlinear Alternative [10,19], which we now state.

THEOREM 1. *Let C be a convex subset of a normed linear space E and assume $0 \in C$. Let $F : C \to C$ be a completely continuous operator, and let*

$$\mathcal{E}(F) = \{x \in C \mid x = \lambda F x \text{ for some } 0 < \lambda < 1\}.$$

Then, either $\mathcal{E}(F)$ is unbounded, or F has a fixed point.

Now, by a solution $x(t)$ of (1)–(3), we mean $x : [-N, T+2] \to \mathbb{R}$, x satisfies (1) on $[1, T+1]$, x satisfies (2) on $[-N, 0]$, and x satisfies (3).

If $G(t,s)$ is the Green's function for

$$\begin{aligned} \Delta^2 x(t-1) &= 0, \\ x(0) = x(T+2) &= 0, \end{aligned}$$

then by [20–24], the boundary value problem (1)–(3) can be transformed into the piecewise defined summation equation,

$$x(t) = \begin{cases} \sum_{s=1}^{T+1} G(t,s) f\left(s, x_{s-1}, \Delta x(s-1)\right) + \dfrac{A - \phi(0)}{T+2} t + \phi(0), & 0 \le t \le T+2, \\ \phi(t), & -N \le t \le 0. \end{cases}$$

Consequently, obtaining solutions of (1)–(3) consists of finding fixed points of the mapping S defined by

$$Sx(t) = \begin{cases} \sum_{s=1}^{T+1} G(t,s) f\left(s, x_{s-1}, \Delta x(s-1)\right) + \dfrac{A - \phi(0)}{T+2} t + \phi(0), & 0 \le t \le T+2, \\ \phi(t), & -N \le t \le 0. \end{cases}$$

Application of Theorem 1 involves *a priori* bounds on a family of eigenfunctions in order to obtain a fixed point.

THEOREM 2. *Let $f : [1, T+1] \times \mathcal{F}_N \times \mathbb{R} \to \mathbb{R}$ be continuous, and assume there exists an $M > 0$ such that, for each $0 \le \lambda \le 1$,*

$$\|x\|_{[0,T+2]} \le M,$$

for all solutions x of the boundary value problem,

$$\begin{aligned} \Delta^2 x(t-1) &= \lambda f\left(t, x_{t-1}, \Delta x(t-1)\right), & t &\in [1, T+1], & (4) \\ x(s) &= \phi(s), & s &\in [-N, 0], & (5) \\ x(T+2) &= \lambda A. & & & (6) \end{aligned}$$

Then, the boundary value problem (1)–(3) has at least one solution.

PROOF. We remark at the outset, for $\lambda = 1$, boundary value problem (4)–(6) is the same as (1)–(3). The proof consists of the two cases, $\phi(0) = 0$ and $\phi(0) \neq 0$.

Assume first that $\phi(0) = 0$. Let the space of functions

$$E = \{x \mid x : [0, T+2] \to \mathbb{R}\}$$

with norm $\|\cdot\|_{[0,T+2]}$. Then, let the convex set C be given by

$$C = \{x \in E \mid x(0) = 0\}.$$

Finally, define an operator $F : C \to E$ by

$$Fx(t) = \sum_{s=1}^{T+1} G(t,s) f\left(x, x_{s-1}, \Delta x(s-1)\right) + \frac{A}{T+2}t, \qquad 0 \le t \le T+2,$$

where

$$x_s(\theta) = \begin{cases} x(s+\theta), & \text{if } s+\theta \ge 0, \\ \phi(s+\theta), & \text{if } s+\theta < 0. \end{cases}$$

It is clear that $F(C) \subseteq C$. That F is continuous is immediate from the continuity of f itself.

To argue that F is compact, let $\{x(k)\}_{k=1}^{\infty}$ be a bounded sequence in C. It follows that there is a subsequence $\{x(k(j))\}$ and some $\bar{x} \in C$ such that $x(k(j))(t) \to \bar{x}(t)$ on $[0, T+2]$. Thus, it readily follows that

$$F\left(x\left(k(j)\right)\right)(t) = \sum_{s=1}^{T+1} G(t,s) f\left(s, x\left(k(j)\right)_{s-1}, \Delta x\left(k(j)\right)(s-1)\right) + \frac{A}{T+2}t$$

converges to $F(\bar{x}(t))$ on $[0, T+2]$. This coupled with the continuity of F implies that F is completely continuous.

Now the hypotheses on solutions of (4)–(6) imply that $\mathcal{E}(F) = \{x \in C \mid x = \lambda F x$, for some $0 < \lambda < 1\}$ is a bounded set. We apply Theorem 1 to obtain a fixed point, $x \in C$, of the operator F. The function,

$$z(t) = \begin{cases} x(t), & t \in [0, T+2], \\ \phi(t), & t \in [-N, 0], \end{cases}$$

is a solution of (1)–(3) for the case $\phi(0) = 0$.

To complete the proof, we assume $\phi(0) \neq 0$. For this case, we make the transformation in (1)–(3),

$$y = x - \phi(0),$$

to obtain the new boundary value problem,

$$\Delta^2 y(t-1) = f\left(t, y_{t-1} + \phi(0), \Delta y(t-1)\right) \equiv \bar{f}\left(t, y_{t-1}, \Delta y(t-1)\right), \qquad t \in [1, T+1], \tag{7}$$

$$y(s) = \phi(s) - \phi(0) \equiv \bar{\phi}(s), \qquad s \in [-N, 0], \tag{8}$$

$$y(T) = A - \phi(0) \equiv \bar{A}, \tag{9}$$

for which $\bar{\phi}(0) = 0$ (here $\equiv$ denotes "defined by"). The first case yields a solution $y(t)$ of (7)–(9). Then $x(t) = y(t) + \phi(0)$ is a solution of (1)–(3). The proof is complete.

APPLICATION. Suppose that $f : [1, T+1] \times \mathcal{F}_N \to \mathbb{R}$ is continuous and that there exists an $M > 0$ such that, for every sequence $y : [0, T+2] \to \mathbb{R}$ and t_0 for which $|y(t_0)| > M$, then $y(t_0) f(t_0, y_{t_0-1}) > 0$. Then, for each $\phi \in \mathcal{F}_N$ with $\phi(0) = 0$, there exists a solution of

$$\Delta^2 y(t-1) = f\left(t, y_{t-1}\right), \qquad t \in [1, T+1], \tag{10}$$

$$y(s) = \phi(s), \qquad s \in [-N, 0], \tag{11}$$

$$y(T+2) = 0. \tag{12}$$

To see this, suppose $y(t)$ is a solution of (10)–(12). If $y(t_0)$ is a positive maximum at some $t_0 \in [1, T+1]$, then $\Delta^2 y(t_0 - 1) \leq 0$, so that $y(t_0) f(t_0, y_{t_0 - 1}) \leq 0$. Consequently, $y(t_0) \leq M$. Similarly, if $y(t_0)$ is a negative minimum at some $t_0 \in [1, T+1]$, then $\Delta^2 y(t_0 - 1) \geq 0$, so that $y(t_0) f(t_0, y_{t_0-1}) \leq 0$. Again, $y(t_0) \geq -M$. Hence, $|y(t)| \leq M$, for all $t \in [0, T+2]$. The same property holds for the appropriate one-parameter family, and Theorem 2 can be applied.

3. SOME GENERALIZATION

There are a number of directions in which Theorem 2 can be generalized. In this section, we state two such generalizations. The first case involves Sturm-Liouville boundary conditions, and the second generalization is for n^{th} order equations. The proofs require the existence of Green's functions and the equivalence of summation equations [18,20–23]. The arguments are completely analogous to those for Theorem 1, and hence we only state these generalizations.

For our first result, let $\phi \in \mathcal{F}_N$ and $A \in \mathbb{R}$ be given. Then, we are concerned with solutions of equation (1) satisfying

$$x(s) = \phi(s), \qquad s \in [-N, 0], \tag{13}$$

and

$$\alpha x(T+1) + \beta \Delta x(T+1) = A, \tag{14}$$

where $\beta \neq 0$.

THEOREM 3. *Let $f : [1, T+1] \times \mathcal{F}_N \times \mathbb{R} \to \mathbb{R}$ be continuous, and assume there exists an $M > 0$ such that, for each $0 \leq \lambda \leq 1$,*

$$\|x\|_{[0,T+2]} \leq M,$$

for all solutions x of the boundary value problem,

$$\Delta^2 x(t-1) = \lambda f\left(t, x_{t-1}, \Delta x(t-1)\right), \qquad t \in [T+1], \tag{15}$$

$$x(s) = \phi(s), \qquad s \in [-N, 0], \tag{16}$$

$$\alpha x(T+1) + \beta \Delta x(T+1) = \lambda A. \tag{17}$$

Then the boundary value problem (1),(10),(11) has at least one solution.

For a generalization to n^{th} order equations, let $\phi \in \mathcal{F}_N$ and $A_1, \dots, A_{n-1} \in \mathbb{R}$ be given. We consider solutions of

$$\Delta^n x(t-1) = f\left(t, x_{t-1}, \Delta x(t-1), \dots, \Delta^{n-1} x(t-1)\right), \qquad t \in [1, T+1], \tag{18}$$

$$x(s) = \phi(s), \qquad s \in [-N, 0], \tag{19}$$

$$\Delta^i x(T+2) = A_{i+1}, \qquad 0 \leq i \leq n-2. \tag{20}$$

THEOREM 4. *Let $f : [1, T+1] \times \mathcal{F}_N \times \mathbb{R}^{n-1} \to \mathbb{R}$ be continuous, and assume there exists $M > 0$ such that, for each $0 \leq \lambda \leq 1$,*

$$\|x\|_{[0,T+n]} \leq M,$$

for all solutions x of the boundary value problem,

$$\Delta^n x(t-1) = \lambda f\left(t, x_{t-1}, \Delta x(t-1), \dots, \Delta^{n-1} x(t-1)\right), \qquad t \in [1, T+1], \tag{21}$$

$$x(s) = \phi(s), \qquad s \in [-N, 0], \tag{22}$$

$$\Delta^i x(T+2) = \lambda A_{i+1}, \qquad 0 \leq i \leq n-2. \tag{23}$$

Then, the boundary value problem (15)–(20) has at least one solution.

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